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Triangles from the Feuerbach Triangle

Trokuti iz Feuerbachovog trokuta

SAŽETAK

U članku dokazujemo četiri neobična teorema o Feuerbachovom trokutu $A_dB_dC_d$ zadanog trokuta ABC kome su vrhovi u točkama A_d, B_d, i C_d gdje pripisane kružnice dotiču izvana kružnicu devet točaka. Ti rezultati odnose se na problem određivanja trokuta XYZ za koje će segmenti A_dX, B_dY, i C_dZ biti stranice trokuta. Pronađeno je pet trokuta XYZ (uključujući i degenerirani trokut u točki D gdje upisana kružnica iznutra dodiruje kružnicu devet točaka) pridruženih trokutu ABC takvih da segmenti A_dX, B_dY, i C_dZ nikada nisu stranice bilo kakvog trokuta. Na pozitivnoj strani, otkrivena su tri trokuta XYZ takva da su segmenti A_dX, B_dY, i C_dZ uvijek stranice nekog trokuta. Dokazi se provode čistom algebarskom metodom upotrebom analitičke geometrije ravnine. Također se pokazuje kako se ti i njima slični rezultati mogu otkriti pomoću računalnog programa Geometer's Sketchpad (Geometrova Crtanka).

Ključne riječi: trokut, upisana kružnica, pripisane kružnice, kružnica devet točaka, Feuerbachov trokut, centralne točke trokuta, Feuerbachova točka, Geometer's Sketchpad

Triangles from the Feuerbach Triangle

ABSTRACT

We prove four unusual theorems about the Feuerbach triangle $A_dB_dC_d$ of the given triangle ABC whose vertices are points A_d, B_d, and C_d in which the excircles touch from outside the nine-point circle. These results concern the problem to determine for which triangles XYZ will the segments A_dX, B_dY, and C_dZ be sides of a triangle. We shall find five triangles XYZ (including the degenerate triangle at the point D in which the incircle touches from inside the nine-point circle) associated to a triangle ABC such that A_dX, B_dY, and C_dZ are never sides of a triangle. On the positive side, we discover three triangles XYZ such that the segments A_dX, B_dY, and C_dZ are always sides of a triangle. We give an algebraic method of proof for these results based on simple analytic geometry in the plane. We also show how one can discover these and other related results using the Geometer's Sketchpad.

Key words: triangle, incircle, excircles, nine-point circle, Feuerbach triangle, central points, Feuerbach point, Geometer's Sketchpad

MSC 2000: Primary 51N20, 51M04, Secondary 14A25, 14Q05

1 Introduction

Recall the construction of a triangle ABC whose sides are three given segments a, b, and c (see Figure 1). First pick a point B in the plane and select a point C on a circle with centre at B and radius a. Then draw circles with centres at B and C and radii c and b. Intersections of these two circles determine two possibilities for the third vertex A. Hence, there is only one solution when we require that going from A to B and then to C is in the counterclockwise direction.

The condition for the existence of solutions is that the inequalities $a < b+c$, $b < c+a$, and $c < a+b$ hold. Since a, b, and c are positive, this condition is equivalent with the requirement that $T[a] > 0$, where $[a]$ is a short notation for the triple (a,b,c) and $T[a]$ is the product $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$ which expands to $2\,(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)$. When it is positive, $T[a]$ is equal to 16 times the square of the area S of ABC.

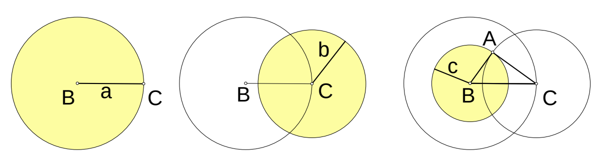

Fig. 1 Construction of a triangle from three segments.

The function T can be therefore utilised to decide when three segments are sides of a triangle. These three segments could be given in myriad of ways. One of the most natural is to take them as segments PX, QY, and RZ joining corresponding vertices of triangles PQR and XYZ (see Figure 2(a)) or as segments PW, QW, and RW joining vertices of a triangle PQR with a point W (see Figure 2(b)). We write $XYZ \in \Omega(PQR)$ and $W \in \Omega(PQR)$ when $T[PX] > 0$ and $T[PW] > 0$, respectively.

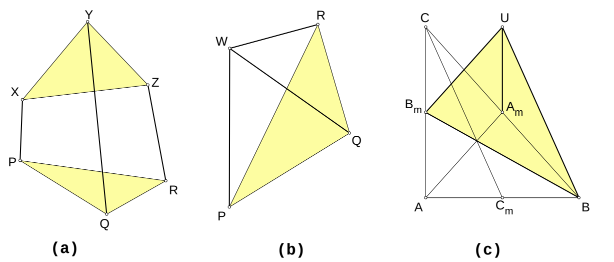

Fig. 2 (a) Triangles from segments joining vertices with vertices.
(b) Triangles from segments joining vertices with a point.
(c) The triangle from the medians.

For example, if A_m, B_m, and C_m are midpoints of sides BC, CA, and AB of the base triangle ABC, then the statement that (the complementary triangle) $A_mB_mC_m$ is in $\Omega(ABC)$ is equivalent to the well-known fact that medians AA_m, BB_m, and CC_m are sides of a triangle.

The simplest proof of $A_mB_mC_m \in \Omega(ABC)$ is based on the Figure 2(c) from [3, p. 282]. The segments A_mU and B_mU are parallel to AB_m and AA_m so that the triangle BB_mU has medians as sides.

Another entirely algebraic proof that is equally simple if we do it with a computer requires first to find lengths of medians AA_m, BB_m, and CC_m and then to show that $T[AA_m] > 0$. Since $2AA_m$ is equal $\sqrt{2b^2+2c^2-a^2}$, and $2BB_m$ and $2CC_m$ are $\sqrt{2c^2+2a^2-b^2}$ and $\sqrt{2a^2+2b^2-c^2}$, we easily find $T[AA_m] = \frac{9}{16}S^2 > 0$.

The present article takes in problems shown in Figures 2(a) and 2(b) for the triangle PQR the Feuerbach triangle $A_dB_dC_d$ of a given triangle ABC and searches for triangles and points in $\Omega(A_dB_dC_d)$ or its complement among various triangles and points naturally associated to ABC. If you wonder what is so special about the Feuerbach triangle, keep in mind that some of the most beautiful theorems in triangle geometry have been proved about it in the last 250 years and that our results below show surprising role of this triangle even in such a basic question as when three segments are sides of a triangle. The surprise comes from the black and white nature of our results: from segments joining vertices of $A_dB_dC_d$ with vertices of some triangles we always get a triangle while there are triangles when we never get a triangle in this way. An interesting recent article about the Feuerbach triangle is [6].

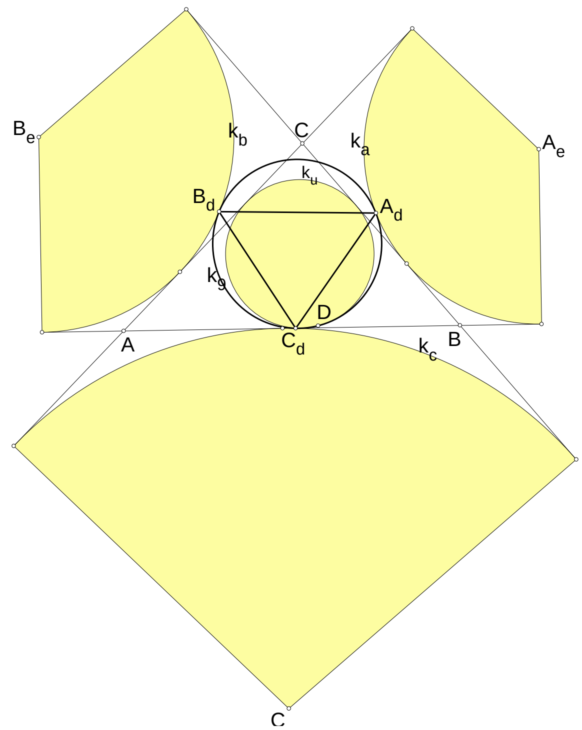

Fig. 3 The nine-point circle is touched from outside by the excircles at vertices of the Feuerbach triangle and from inside by the incircle at the Feuerbach point of ABC.

The vertices A_d, B_d, and C_d of the Feuerbach triangle are points in which the excircles k_a, k_b, and k_c touch from outside the circumcircle k_9 of the complementary triangle $A_mB_mC_m$. The circle k_9 is also known as the nine-point circle because it goes through the midpoints A_m, B_m, and C_m of sides, the feet A_o, B_o, and C_o of altitudes, and the midpoints A_f, B_f, and C_f of segments joining the orthocentre H (concurrence point of altitudes) with vertices A, B, and C.

The above statement about excircles touching the nine-point circle from outside is just a part of the famous Feuerbach theorem from 1834 which also established that the incircle k_u makes a touch with the nine-point circle from inside at the so called Feuerbach point D of ABC (see [3] and Figure 3 above).

2 Statement of Theorems

In order to describe our main results we need the triangle $A_pB_pC_p$ at points of contact of the incircle with sides of ABC and the triangle $A_nB_nC_n$ at points of intersection of internal angle bisectors with sides of ABC (i. e., the intouch triangle (cevian triangle of the Gergonne point) and the incentral triangle (cevian triangle of the incentre)).

Theorem 1 *In every triangle ABC it is impossible to construct a triangle from segments A_dD, B_dD, and C_dD joining the vertices of its Feuerbach triangle with its Feuerbach point.*

In our notation, this theorem simply claims that $D \notin \Omega(ABC)$ holds for every triangle ABC. The next theorem is a similar statement for four triangles.

Let A_g, B_g and C_g denote points diametrically opposite on the nine-point circle to the points A_f, B_f and C_f.

Theorem 2 *In every triangle ABC it is impossible to construct a triangle from segments A_dX, B_dY, and C_dZ when XYZ is $A_xB_xC_x$ for $x \in \{g, m, n, p\}$.*

The next two theorems are different because they describe situations when segments to vertices of the Feuerbach triangle are always sides of a triangle.

Theorem 3 *If the triangle ABC is not isosceles, then the segments A_dA_o, B_dB_o, and C_dC_o joining the vertices of its Feuerbach triangle with the vertices of its orthic triangle $A_oB_oC_o$ are sides of a triangle.*

Let A_q, B_q and C_q denote points diametrically opposite on the nine-point circle to the midpoints A_m, B_m and C_m of sides.

Theorem 4 *In every triangle ABC, the segments joining the vertices of its Feuerbach triangle $A_dB_dC_d$ with the vertices of either its Euler triangle $A_fB_fC_f$ or the antipodal $A_qB_qC_q$ of its complementary triangle $A_mB_mC_m$ are sides of an acute triangle.*

3 Theorems and the Geometer's Sketchpad

In this section we shall explain how one can discover and check our theorems using the computer software the Geometer's Sketchpad. This program allows one to explore properties of geometric objects and constructions in a dynamical fashion because it remembers relationships and readjusts all calculations as you move objects around (on the screen).

Let us first describe how to make a script `test.gss` which will test if segments PX, QY, and RZ from Figure 2(a) are sides of a triangle. For this one must draw this figure, measure lengths of these segments, and calculate $T[PX]$. Hide everything except the vertices and the calculation, select with Shift key all visible objects (one can do this also with Ctrl+a), and then use Work menu to make the script. In applying script `test.gss` one must select six points in correct order. Its action will give a value of the function T for segments joining corresponding vertices of two triangles. As we move points around we must look whether this value is positive. Then the segments are sides of a triangle. On the other hand, when this value is zero or negative, then the segments are not sides of a triangle. The same test applies to the situation of Figure 2(b). Simply select the point W three times.

The next task is to make scripts for all points and triangles which appear in our theorems. Of course, some of them, like `h.gss` for the orthocentre of a triangle or `tr_m.gss` for the complementary triangle of a triangle, are straightforward. Others, like `d.gss` for the Feuerbach point of a triangle and `tr_d.gss` for the Feuerbach triangle of a triangle, are a bit tricky. It would be wrong to use Figure 3 and the Feuerbach theorem because the Geometer's Sketchpad has difficulties in finding intersections of two circles and for it touching point is always a pair of points.

A way out from these difficulties is to construct these points from their trilinear coordinates i. e., from any triple of real numbers proportional to their distances from sidelines of the base triangle. We shall illustrate this by describing the script `d.gss` for the Feuerbach point D. It is well-known (see references [4] and [2] and section 6 below) that D has trilinears $t_a = 1 - cos(B - C)$, $t_b = 1 - cos(C - A)$, and $t_c = 1 - cos(A - B)$ and that the actual distance of D from BC is $d_a = 2St_a/(t_aBC + t_bCA + t_cAB)$ while d_b and d_c have similar expressions.

Draw a triangle ABC and rotate points C and A around points B and C for 90 degrees in the counterclockwise direction to get points C' and A'. Calculate sides BC, CA, and AB, angles A, B, and C, and distances d_a and d_b. Then dilate points C' and A' with respect to centres B and C for marked ratios $\lambda d_a/BC$ and $\lambda d_b/CA$ to get points C'' and A'', where λ is the product of signs of angles A, B, and C. Finally, D will be the intersection of parallels through C'' and A'' to BC and CA (see Figure 4). Hide everything except A, B, C, and D, select with Shift these four points, and use the Work menu to make the script. Its action produces the Feuerbach point of a triangle whose vertices have been selected.

For the Feuerbach triangle we must know that trilinears of A_d are $-\sin^2(\frac{B-C}{2})$, $\cos^2(\frac{C-A}{2})$, and $\cos^2(\frac{A-B}{2})$, and that trilinears of B_d and C_d are their cyclic permutations. In section 5 below we shall show how one can compute these trilinears.

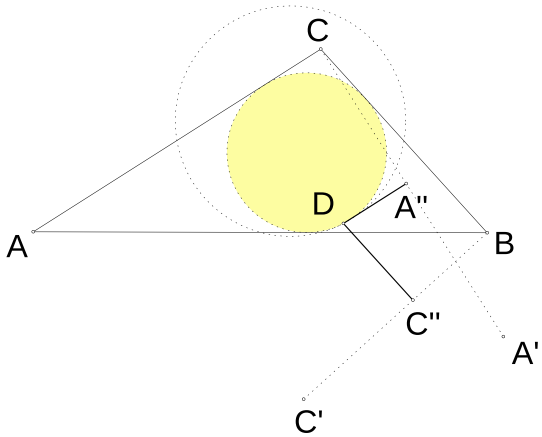

Fig. 4 Sketch explaining the script `d.gss` for the Feuerbach point of a triangle.

How does one check the claim $D \notin \Omega(ABC)$ from the Theorem 1? Well, draw ABC, use `tr_d.gss` and `d.gss` to get its Feuerbach triangle $A_dB_dC_d$ and its Feuerbach point D, and then apply `test.gss` to compute the triangle test for the triple $[A_dD]$. As we move the point C around the value of the test is never positive so that these segments are never sides of a triangle. The arguments for other our claims are analogous.

Of course, computer software like the Geometer's Sketchpad can only help us to discover theorems and quickly eliminate false conjectures but they can not give us (for now) mathematically sound proofs. For one thing, the Geometer's Sketchpad has limited precision so that no point is determined precisely. Therefore, how can we expect to prove something about the Feuerbach triangle when we don't even know the position of its vertices?

This is a nice example showing the need for rigour in mathematical proofs and for work we are going to do below in proving our theorems. Our idea is to use analytic geometry in the plane and position the base triangle in the coordinate system so that most calculations are rather simple especially when done with software using symbolic computation (like Maple, Mathematica, and Derive) which is nowadays quite common.

4 Placement of *ABC*

We shall position the triangle ABC in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex A is the origin with coordinates $(0,0)$, the vertex B is on the x-axis and has coordinates $(rh,0)$, and the vertex C has coordinates $(gqr/k, 2fgr/k)$, where $h = f+g$, $k = fg-1$, $p = f^2+1$, $q = f^2-1$, $s = g^2+1$, $t = g^2-1$, $u = f^4+1$, $v = g^4+1$, and $w = f-g$. The three parameters r, f, and g are the inradius and the cotangents of half of angles at vertices A and B. Without loss of generality, we can assume that both f and g are larger than 1 (i. e., that angles A and B are acute).

Nice features of this placement are that most important points related to the triangle ABC (including all central points from Table 1 in [4]), have rational functions in f, g, and r as coordinates and that we can easily switch from f, g, and r to side lengths a, b, and c and back with substitutions

$$a = \frac{rfs}{k}, \qquad b = \frac{rgp}{k}, \qquad c = rh,$$

$$f = \frac{(b+c)^2-a^2}{\sqrt{T([a])}}, \; g = \frac{(a+c)^2-b^2}{\sqrt{T([a])}}, \; r = \frac{\sqrt{T([a])}}{2(a+b+c)}.$$

Moreover, since we use the Cartesian coordinate system, computation of distances of points and all other formulas and techniques of analytic geometry are available and well-known to widest audience. A price to pay for these conveniences is that symmetry has been lost.

The third advantage of the above position of the base triangle is that we can easily find coordinates of a point with given trilinears. More precisely, if a point P with coordinates x and y has projections P_a, P_b, and P_c onto the side lines BC, CA, and AB and $\lambda = PP_a/PP_b$ and $\mu = PP_b/PP_c$, then

$$x = \frac{gh(p\mu+q)r}{fs\lambda\mu+gp\mu+hk}, \qquad y = \frac{2fghr}{fs\lambda\mu+gp\mu+hk}.$$

This formulas will greatly simplify our exposition because there will be no need to give explicitly coordinates of points but only its trilinear coordinates. For example, the centre A_e of the A-excircle k_a obviously has trilinears $-1:1:1$. Then we use the above formulas with $\lambda = -1$ and $\mu = 1$ to get the coordinates $(rfgh/k, rgh/k)$ of A_e in our coordinate system.

5 Computation of coordinates of points

In this section we shall explain how to compute coordinates of all points from statements of our theorems.

Let I be the incenter of ABC. Then the inner angle bisectors AI and BI and the external angle bisector at the vertex B have equations

$$\mathbf{e}_1 : \; x - fy = 0, \qquad \mathbf{e}_2 : \; x + gy - hr = 0,$$

$$\mathbf{e}_3 : \; gx - y - ghr = 0.$$

The solution of equations $\mathbf{e}_1$ and $\mathbf{e}_2$ will give us coordinates (fr, r) of the incenter I while the solution of $\mathbf{e}_1$ and $\mathbf{e}_3$ determines the coordinates $A_e(fghr/k, ghr/k)$ of the center of the A-excircle. Hence, the equations of the incircle and the A-excircle are

$$\mathbf{e}_4 : (x - f\,r)^2 + (y - r)^2 = r^2,$$

$$\mathbf{e}_5 : \left(x - \frac{f\,g\,h\,r}{k}\right)^2 + \left(y - \frac{g\,h\,r}{k}\right)^2 = \left(\frac{g\,h\,r}{k}\right)^2.$$

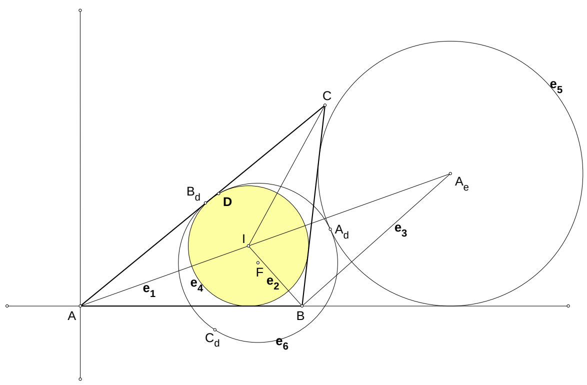

Fig. 5 Lines and circles used for the computation of coordinates.

On the other hand, since the midpoints of sides have coordinates

$$A_m\left(\frac{r(f\,s + 2\,g\,q)}{2\,k}, \frac{f\,g\,r}{k}\right),$$

$$B_m\left(\frac{g\,q\,r}{2\,k}, \frac{f\,g\,r}{k}\right), \qquad C_m\left(\frac{h\,r}{2}, 0\right),$$

it is easy to compute coordinates

$$F\left(\frac{r\,(f\,t + 3\,g\,q)}{4\,k}, \frac{r\,((k+2)^2 - w^2)}{8\,k}\right)$$

of the center of the nine-point circle and its radius $\frac{p\,r\,s}{8\,k}$. It follows that the nine-point circle has the equation

$$\mathbf{e}_6 : \left(x - \frac{r\,(f\,t + 3\,g\,q)}{4\,k}\right)^2 + \left(y - \frac{r\,((k+2)^2 - w^2)}{8\,k}\right)^2 = \left(\frac{p\,r\,s}{8\,k}\right)^2.$$

The equations $\mathbf{e}_5$ and $\mathbf{e}_6$ have only one solution which determines coordinates $\left(\frac{u_a}{k\,m_a}, \frac{v_a}{k\,m_a}\right)$ of the touching A_d of the A-excircle and the nine-point circle, where $u_a = g\,h\,r\,(f\,q\,s + 6\,g\,q + 4\,f\,g^2)$, $v_a = 2\,g\,h\,r\,(k+2)^2$, and $m_a = p\,s + 8\,(k+s)$. In an analogous way one can compute coordinates $\left(\frac{u_b}{k\,m_b}, \frac{v_b}{k\,m_b}\right)$ and $\left(\frac{u_c}{m_c}, \frac{v_c}{m_c}\right)$ of the other two vertices B_d and C_d of the Feuerbach triangle, where $u_b = h\,r\,(q\,s + 6\,f\,g\,q - 4\,f^2)$, $v_b = 2\,f\,h\,r\,(k+2)^2$, $m_b = p\,s + 8\,(k+q)$, $u_c = g\,r\,(3\,q\,s + 6\,f\,g\,q + 4\,g^2 - 4\,q)$, $v_c = -2\,f\,g\,r\,w^2$, and $m_c = p\,s + 8\,f\,g\,k$.

Similarly, the equations $\mathbf{e}_4$ and $\mathbf{e}_6$ have only one solution which determines coordinates $\left(\frac{u_0}{m_0}, \frac{v_0}{m_0}\right)$, of the touching point D of the nine-point circle and the incircle, where $u_0 = r\,(f\,q\,s - 6\,g\,q + 4\,f)$, $v_0 = 2\,r\,w^2$, and $m_0 = p\,s - 8\,k$.

We can now compute the trilinear coordinates of D as follows. The third trilinear coordinate of D is proportional to v_0/m_0 (the ordinate of D is its distance from AB). On the other hand,

$$1 - \cos(A - B) = 2\,\sin^2\left(\frac{A-B}{2}\right) = \frac{2\,(B^* - A^*)^2}{(B^* - A^*)^2 + (1 + B^* A^*)^2} = \frac{2\,w^2}{p\,s},$$

where $A^* = \cot(A/2) = f$ and $B^* = \cot(B/2) = g$. But, $p\,s/m_0$ represented in terms of a, b, and c is easily seen to be symmetric, so that the third trilinear of D is indeed $1 - \cos(A - B)$. The other two are $1 - \cos(B - C)$ and $1 - \cos(C - A)$. We can verify this using the transfer formulas from trilinears to our coordinates.

6 Proof of $D \notin \Omega(A_d B_d C_d)$

With the standard formula $d(P, Q) = \sqrt{(p-q)^2 + (x-y)^2}$ for the Euclidean distance between points $P\,(p, x)$ and $Q\,(q, y)$, we find

$$A_d D = \frac{p\,r\,s\,|i_a|}{k\sqrt{m_0\,m_a}}, \quad B_d D = \frac{p\,r\,s\,|i_b|}{k\sqrt{m_0\,m_b}}, \quad C_d D = \frac{p\,r\,s\,|w|}{\sqrt{m_0\,m_c}},$$

for distances from points A_d, B_d, and C_d to the point D, where $i_a = f\,t - 2\,g$ and $i_b = g\,q - 2\,f$. Hence, $T[A_d D] = -32\,p^4\,r^4\,s^4\,w^2\,i_a^2\,i_b^2\,m_1/(k^4\,m_a^2\,m_b^2\,m_c^2)$, where m_1 is the following polynomial

$$(k+2)(2\,k+1)\,h^6 + k\,(4\,k^3 + 43\,k^2 + 68\,k + 32)\,h^4 + k^2\,(2\,k^2 + 11\,k + 8)(k^2 + 14\,k + 16)\,h^2 + k^7.$$

Since the expression m_1 is clearly positive (recall that $h > 2$ and $k > 0$), we conclude that $T[A_d D]$ is never positive so that the segments $A_d D$, $B_d D$, and $C_d D$ can not be sides of a triangle regardless of the shape of the triangle ABC.

7 Proof of $A_mB_mC_m \notin \Omega(A_dB_dC_d)$

In the same way as above we can compute distances of points A_d, B_d, and C_d from the midpoints of sides A_m, B_m, and C_m.

$$A_dA_m = \frac{|i_a|\, m_2}{k\sqrt{m_a}}, \quad B_dB_m = \frac{|i_b|\, m_2}{k\sqrt{m_b}},$$

$$C_dC_m = \frac{|w|\, m_2}{\sqrt{m_c}}, \quad m_2 = \frac{r\sqrt{ps}}{2}$$

and find $T[A_dA_m] = -2\, p^2 r^4 s^2 w^2 i_a^2 i_b^2 m_3/(k^4 m_a^2 m_b^2 m_c^2)$, where m_3 is the polynomial

$$\begin{aligned} & k(2k-3)\,h^6 \\ & +(k-2)(4k^3+19k^2-56k+36)\,h^4 \\ & \quad +(k-2)^2(2k^2+3k-6)(k^2+10k-8)\,h^2+(k-2)^7. \end{aligned}$$

Since m_3 becomes a polynomial with all coefficients positive after the substitutions $f = 1+f'$ and $g = 1+g'$ (recall that $f > 1$ and $g > 1$ and thus $f' > 0$ and $g' > 0$), we conclude that $T[A_dA_m]$ is never positive so that the segments A_dA_m, B_dB_m, and C_dC_m also can not be sides of a triangle for any triangle ABC.

8 Remarks on proofs of remaining cases

The proofs of the remaining six cases are almost identical to the case with midpoints of sides. The only difference is that polynomials corresponding to the polynomial m_3 become far more complicated and difficult to write down. This is even more so for polynomials that we obtain after the above substitutions because they have hundreds (up to 570) terms. It is now clear that our method of proof is almost impossible without use of computers. Also, in order to check our claims in the rest of the paper, the reader should make a try with some package for symbolic computation (like Maple, Mathematica, or Derive). We shall only give some expressions that can serve as pointers to all those who will attempt such a work-out. Therefore, this paper is an example of a new type of articles in mathematics which can be fully appreciated only by those readers that are willing to read it interactively. The standards for exposition of such papers is only emerging so that our presentation might appear unusual or inadequate to some readers.

The author does not rule out the possibility that our results have much simpler proofs with traditional geometric methods. Hence, our paper and its approach might challenge readers to think of such old-fashioned proofs for Theorems 1 - 4.

9 Proof of Theorem 3

Points A_o, B_o, and C_o are projections of the vertices A, B, and C onto the opposite sides BC, CA, and AB. Hence, we get

$$A_o\left(\frac{4g^2hr}{s^2}, \frac{2ghrt}{s^2}\right),$$

$$B_o\left(\frac{hq^2r}{p^2}, \frac{2fhqr}{p^2}\right), \qquad C_o\left(\frac{gqr}{k}, 0\right).$$

The distances of points A_d, B_d, and C_d from the points A_o, B_o, and C_o are

$$A_dA_o = \frac{ghr|ft-2g|\sqrt{p}}{k\sqrt{sm_a}}, \quad B_dB_o = \frac{fhr|gq-2f|\sqrt{s}}{k\sqrt{pm_b}},$$

$$C_dC_o = \frac{fgr|w|\sqrt{ps}}{k\sqrt{m_c}},$$

Then $T[A_dA_o] = r^4 w^2 i_a^2 i_b^2 m_4/(k^4 p^2 s^2 m_a^2 m_b^2 m_c^2)$, where $m_4 = \sum_{i=0}^{7} k_i k^{\lambda_i} h^{2i}$ with $\lambda_i = 12, 8, 4, 2, 0, 0, 0, 0$ for $i = 0, 1, \ldots, 7$ and k_i is a (product of) polynomial(s) in the variable k represented as sequences $(a_0, \ldots, a_n)$ of their integer coefficients as follows:

k_0	$4(1,1)^3$
k_1	$(512, 1024, 600, 136, 35)$
k_2	$2\,(1,1)(8192, 32768, 52224, 44032, 23422, 9332, 2631, 331)$
k_3	$(32768, 229376, 666624, 1056768, 1002704, 583760, 203133, 38174, 2845)$
k_4	$(16384, 147456, 563200, 1200128, 1564604, 1277204, 636048, 176284, 20820)$
k_5	$(512, 6144, 22360, 37624, 33077, 15050, 2845)$
k_6	$(452, 1916, 3150, 2364, 662)$
k_7	$(5,7)\,(7,5)$.

For example, polynomials k_0 and k_6 are $4\,(k+1)^3$ and $662\,k^4+2364\,k^3+3150\,k^2+1916\,k+452$.

The above is the first example of our method of writing down in compact form rather lengthly polynomials like m_4. We simply write in parenthesis their coefficients in the increasing order starting with the trailing coefficient and since ours are polynomials in variables h and k we give polynomials of k as coefficients of powers of h.

The polynomial m_4 has all coefficients positive so that the triangle test $T[A_dA_o]$ is always positive unless ABC is isosceles when it is zero. It follows that segments A_dA_o, B_dB_o, and C_dC_o will be sides of a triangle for any triangle ABC which is not isosceles.

10 Proof of $A_fB_fC_f \in \Omega(A_dB_dC_d)$

Points A_f, B_f, and C_f are midpoints of the segments AH, BH, and CH, where H is the orthocenter of ABC. Since H has coordinates $(gqr/k, qrt/2k)$, we get

$$A_f\left(\frac{gqr}{2k}, \frac{grt}{4k}\right),\; B_f\left(\frac{r(2gq+ft)}{2k}, \frac{qrt}{4k}\right),$$

$$C_f\left(\frac{gqr}{k}, \frac{r((k+2)^2-w^2)}{4k}\right).$$

Hence, the distances A_dA_f, B_dB_f, and C_dC_f are $m_5\,m_6/\sqrt{m_a}$, $m_5\,m_7/\sqrt{m_b}$, and $m_5\,m_8/\sqrt{m_c}$, where
$m_5 = r\sqrt{ps}/(4k)$, $m_6 = qs+4fg+4g^2$,
$m_7 = pt+4fg+4f^2$, and $m_8 = qt+4f^2g^2$.

We obtain $T[A_dA_f] = p^2r^4s^2m_9/(256k^4m_a^2m_b^2m_c^2)$, where $m_9 = \sum_{i=0}^{8} k_i\,k^{\lambda_i}\,h^{2i}$ with $\lambda_i = 11, 6, 4, 2, 1, 0, 0, 0, 0$ for $i = 0, \ldots, 8$ and

$\boxed{k_0}$ $(4,3)^2(128,256,152,27)$

$\boxed{k_1}$ $8\,(32768,172032,377344,454400,332544,154112,44938,7632,585)$

$\boxed{k_2}$ $4(131072,966656,2772992,4162560,3627008,1897856,589452,100840,7341)$

$\boxed{k_3}$ $8\,(32768,319488,1248256,2487808,2785280,1809408,670562,130568,10415)$

$\boxed{k_4}$ $2\,(32768,434176,1930240,3859968,3956992,2140648,572816,58393)$

$\boxed{k_5}$ $(4096,120832,706560,1445888,1323504,550912,83320)$

$\boxed{k_6}$ $(6144,66048,142608,112992,29364)$

$\boxed{k_7}$ $(2736,7488,4680)$

$\boxed{k_8}$ (243).

The polynomial m_9 has all coefficients positive so that the triangle test $T[A_dA_f]$ is always positive. It follows that segments A_dA_f, B_dB_f, and C_dC_f will be sides of a triangle for any triangle ABC.

In order to show that the triangle with sides A_dA_f, B_dB_f, and C_dC_f is acute recall that the triangle ABC is acute, right, or obtuse if and only if the product $U[a] = (b^2+c^2-a^2)(a^2-b^2+c^2)(a^2+b^2-c^2)$ is positive, zero, or negative (see [1]). Here, this product is $U[A_dA_f] = p^4r^6s^4m_{10}\,m_{11}\,m_{12}/(4096k^6m_a^3m_b^3m_c^3)$, where the polynomials m_{10}, m_{11}, and m_{12} become polynomials in f' and g' with all coefficients positive after the substitution $f = 1+f'$ and $g = 1+g'$.

11 Other proofs and extensions

We leave proofs of $A_qB_qC_q \in \Omega(A_dB_dC_d)$ and $A_xB_xC_x \notin \Omega(A_dB_dC_d)$ for $x = p, n, g$ to the reader because they are almost identical to the above proofs. The point here is not that our method is elegant or simple (in the traditional sense), but that the same method applies to all cases.

Another method of proof of our theorems is to express everything in terms of the side lengths a, b, and c. For the Theorem 1 and the part of Theorem 2 for $A_mB_mC_m$, the triangle test is easily seen to be always negative. For other cases the procedure is to write the numerator of the triangle test in terms of the three basic symmetric polynomials in variables a, b, and c and then use the fact [5, p. 7] that they are roots of the polynomial $x^3-2\sigma x^2+(\sigma^2+r(r+4R))x-4rR\sigma$, where σ is the semi-perimeter, r is the inradius, and R is the circumradius. In this way we obtain a polynomial in σ with coefficients polynomials in r and R. Now using the Euler inequality $R \geq 2r$ and the fundamental inequalities between σ, r, and R (see [5, Chapter I]) in each case we can argue that the triangle test function is either always positive or is never positive. However, without some help from computers this approach is also difficult.

We close with the following claims which are possible projects from geometry of triangles. Our method with polynomials applies here too.

When ABC is not equilateral, then the centre of the Kiepert hyperbola [2] has the same property as D in Theorem 1.

When ABC is not quite special, then the centre of the Jarabek hyperbola, which goes through the vertices A, B, C, the orthocentre H, and the circumcentre O has the same property as D in Theorem 1.

When ABC is not isosceles, then the segments DA_p, DB_p, and DC_p are always sides of an obtuse triangle.

When ABC does not have angles of either $\pi/3$ or $2\pi/3$ radians, then the segments DA_f, DB_f, and DC_f are always sides of a triangle.

The segments XA_d, YB_d, and ZC_d are never sides of a triangle, where X, Y, Z are second intersections of DA_d, DB_d, and DC_d with the incircle. The same is true for segments XD, YD, and ZD.

An interesting project is to decide which central points X of the triangle ABC have the property that the segments XA_d, XB_d, and XC_d are always (never) sides of a triangle.

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